

# Consistency Conditions for Branes at Orbifold Singularities

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We discuss consistency conditions for branes at orbifold singularities. The conditions have a world-sheet interpretation in terms of tadpole cancellation and a space-time interpretation in terms of anomalies. As examples, we consider type II and type I branes on  $\mathbb{C}^2/\mathbb{Z}_M$  orbifolds. We give orientifold constructions of phases of type I or heterotic string theory, involving branches with extra tensor multiplets, which arise when small  $SO(32)$  instantons sit on orbifold singularities.

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## 1. Introduction

The familiar miracle of string theory is that phenomena on the world-sheet carry over to space-time. In particular, consistency of the theory on the world-sheet implies that the space-time theory is free of anomalies. Our interest here will be relating world-sheet consistency conditions to space-time consistency conditions for quantum field theories living inside D branes. (See [1] for a recent review with references on D branes.) The QFTs living in the world-volume of D branes can be considered without having to include gravity and other stringy modes by taking the  $M_p \rightarrow \infty$  limit. Though the world-volume QFTs may need more data, such as string theory, to obtain a sensible theory in the ultra-violet, at long distances they must be sensible on their own. In particular, as gauge anomalies only depend on the massless spectrum, the world-volume QFTs must be free of gauge anomalies.

For example, in [2] it was argued that an  $SO(32)$  heterotic or type I instanton of zero size, which is a D5 brane, has a supersymmetric (with 8 super-charges)  $Sp(1) \cong SU(2)$  gauge theory with  $N_f = 16$  fundamental flavors living in its world-volume. With  $K$  such D5 branes on top of each other, the 6d world-volume gauge theory becomes  $Sp(K)$  with  $N_f = 16$  matter fields in the  $\square$  and one in the  $\square$ . When the  $SO(32)$  heterotic or type I string theory is compactified to six dimensions on  $K3$ , there is a restriction that  $K \leq 24$  since the total number instantons, large and small, must be 24 in order to have zero  $H$  charge on the compact  $K3$ . But for the uncompactified string theories, the number  $K$  of D5 branes is completely arbitrary – their  $H$  flux can go off to infinity. Therefore, for any  $K$ , the above  $Sp(K)$  gauge theory must be free of 6d gauge anomalies; this is, indeed, true. It would not have been true for  $SO(N)$  with  $N \neq 32$ .

World-sheet tadpole consistency conditions were discussed in [3] in the context of compactification of type I string theory on a  $K3$  realized as a  $T^4/\mathbb{Z}_2$  orientifold. These conditions ensure that the theory living in the uncompactified six dimensions is free of all gauge and gravitational anomalies. In [4], type II and type I string theory was considered on non-compact  $\mathbb{R}^4/\mathbb{Z}_M$  orbifolds and orientifolds. It was pointed out there that the tadpole consistency conditions can be ignored in the non-compact context because such conditions give, for example, a total charge  $\int dH$  which only needs to vanish on a compact space. It was also pointed out in [4] that this is not completely satisfactory as, for example, the type I anomaly condition which requires  $SO(32)$  on  $\mathbb{R}^{10}$  comes from a tadpole condition which is not a source for any physical field.

More generally, on physical grounds, some consistency conditions can be ignored in non-compact cases, corresponding to the fact that fluxes and gravity can leak out into the non-compact dimensions. Other consistency conditions, on the other hand, must apply even in the non-compact context. For example, even in the non-compact context, there must be consistency conditions which ensure that the QFTs living in the world-volume of D branes are anomaly free. These anomalies can not flow out of the branes into the non-compact dimensions.

Our main point is that this distinction between space-time conditions which can and cannot be ignored in the non-compact context is perfectly matched by world-sheet considerations. Keeping the dependence on the compactification volume  $\mathcal{V}$ , certain tadpoles are inversely proportional to  $\mathcal{V}$  and thus automatically vanish, without imposing any conditions, in the  $\mathcal{V} \rightarrow \infty$  limit. Precisely these tadpoles are associated with space-time conditions which no longer need apply when flux or gravity can leak off into the non-compact dimensions. Other tadpoles, on the other hand, do not automatically vanish in the  $\mathcal{V} \rightarrow \infty$  limit and thus can not be ignored in the non-compact context. These tadpoles give precisely the space-time anomaly conditions which must be satisfied even in the non-compact context.

As examples, we extend the analysis of [4] of branes on (possibly blown up) orbifold singularities  $\mathbb{C}^2/\mathbb{Z}_M$  by pointing out certain tadpole consistency conditions and their relation to anomaly conditions in the world volume of the branes. Our consistency relations will be for  $p$  and  $p+4$  branes on the orbifold spaces. In the type IIA context, for example, we can take  $p = 0, 2, 4$ ; for type IIB,  $p = 1, 3$ ; for type I,  $p = 5$ .

In the next section, we discuss the world-sheet tadpole consistency conditions and their relation to space-time conditions. In sect. 3, we consider type II on the  $\mathbb{Z}_M$  orbifold singularity. We complete the observation of [4] that the hyper-Kähler quotient theories of [5] arise physically in the world-volume of branes at the orbifold singularity by showing that tadpole consistency conditions give a relation of [5] between the numbers of colors and flavors of the gauge theory. By noting the mixing between a  $U(1)$  field strength and the NSNS  $B$  field, this relation allows us to derive the value of the  $B$  field for the orbifold; we find agreement with results obtained in [17] by another method.

In sect. 4 we summarize the space-time analysis of [6] for the 6d theories living in the world-volume of type I small instantons at  $\mathbb{C}^2/\mathbb{Z}_M$  singularities. In particular, we review the argument, based on space-time anomalies, that there is a “Coulomb branch<sup>1</sup>” in which

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<sup>1</sup> In terminology following the discussion in [18].

there are specific gauge theories, along with specific numbers of tensor multiplets, living in the D5 branes at orbifold singularities. In sect. 5 we present the orientifold construction of type I on the orbifold singularities. We verify that the tadpole anomalies which remain relevant, even though  $\mathbb{C}^2/\mathbb{Z}_M$  is of infinite volume, reproduce the space-time anomaly conditions found in [6]. We also verify that the orientifold construction automatically gives exactly the right number of tensors for cancelling the reducible part of the gauge anomaly. Thus orientifolds give a definite construction of the “Coulomb branch” phases of [6].

Indeed, we point out that orientifold constructions *necessarily* give phases with extra tensor multiplets<sup>2</sup>. The single exception to this is the original example of [8,3]. In other words, orientifolds never realize standard perturbative type I theories at ALE singularities. Indeed, all other examples of type I orientifolds contained extra tensors; see, for example [8-16]. The interpretation of the extra tensor multiplets is briefly discussed in sect. 6.

## 2. Tadpoles and space-time consistency conditions

Our results indicate that the irreducible parts of the gravitational and gauge anomalies of the six dimensional theories on D5 branes are related to the one loop tadpole anomalies arising from the divergent part of the Klein bottle, Mobius strip, and cylindrical amplitudes of the string theory on the orbifold. We illustrate this relation in the following sections. A similar relation was previously found [16] on compact orbifolds arising from F theory.

Remembering that strings in directions transverse to a brane have Dirichlet boundary conditions, which allow for winding, while those tangent to the brane have Neumann boundary conditions, which allow for momentum, we can determine the volume dependence of the various tadpoles. Let  $\mathcal{V}_4$  be the volume of the four dimensional orbifold, in our case  $\mathbb{C}^2/\mathbb{Z}_M$ ,  $\mathcal{V}_{p+1}$  be the spacetime volume of the  $p$  brane, and  $\mathcal{V}_{5-p}$  the volume transverse to these two spaces. There are generally three types of tadpole anomalies [3]: the untwisted  $p+4$  brane anomalies, with dependences on volumes  $\mathcal{V}_4\mathcal{V}_{p+1}/\mathcal{V}_{5-p}$ ; the untwisted  $p$  brane anomalies, with dependence on volumes  $\mathcal{V}_{p+1}/\mathcal{V}_4\mathcal{V}_{5-p}$ ; and the twisted  $p$  brane anomalies, with dependence on volumes  $\mathcal{V}_{p+1}/\mathcal{V}_{5-p}$ . In the limit where  $\mathcal{V}_4 \rightarrow \infty$ , the untwisted  $p$ -brane

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<sup>2</sup> See, however, [7] for exotic examples with varying numbers of tensor multiplets, including one with no tensor multiplets at all – not even the usual one from the gravity multiplet, which usually contains the dilaton.

anomaly automatically vanishes and can, thus, be ignored. This anomaly determines the number of  $p$  branes (small instantons) and will always be a free parameter in our models.

In the type I case  $p = 5$ , and the untwisted nine-brane anomaly will always determine the number of nine-branes to be 32. There is no freedom here because flux has nowhere to go since  $\mathcal{V}_{5-p}$  is trivial. The twisted tadpole conditions precisely ensure that the 6d gauge theories living in the world-volume of the five-branes are free of spacetime gauge anomalies. In fact, as we illustrate in sect. 5, it is possible to extract (up to constants) the individual irreducible  $\text{tr}F^4$  and the reducible  $\text{tr}F^2\text{tr}F^2$  anomaly terms from the tadpole terms.

In type IIB theories, anomaly cancellation does not allow for any nine-branes. However,  $p + 4(p)$  brane theories with  $p < 5$  will have no spacetime anomalies. The string theory still senses the ten dimensional origin of the theories in that tadpoles can not be truly ignored until the dimension of  $\mathcal{V}_{5-p}$  is large enough so that all flux can escape to infinity. In analogy with electrodynamics, we would not expect this to occur until  $p \leq 2$ . Although it is not associated with an anomaly in the type IIB case, we can constrain the theories by first solving the twisted tadpole conditions for  $\mathcal{V}_{5-p}$  finite. We then assume that, in the limit that all flux can escape to infinity, these conditions are still valid. This assumption is perhaps not entirely justified but is supported by the fact, shown in the next section, that it gives the correct instanton moduli space.

### 3. Type II on a $\mathbb{C}^2/\mathbb{Z}_M$ orbifold

#### 3.1. General aspects

$N$  coincident type II  $p + 4$  branes have a supersymmetric  $U(N)$  gauge theory living in their world-volume [19].  $K$  coincident type II  $p$  branes living inside of this system have a supersymmetric (with eight super-charges)  $U(K)$  gauge theory living in their world volume with  $N$  matter hypermultiplets in the  $\mathbf{K}$  and one in the adjoint  $\mathbf{K}^2$ . This configuration of branes has the interpretation as  $K$   $U(N)$  instantons and, indeed, this  $U(K)$  world-volume theory has a Higgs branch which gives the hyper-Kahler quotient construction of the moduli space of  $K$   $U(N)$  instantons [20].

We now consider the  $p$  branes on a possibly blown-up singularity  $X \cong \mathbb{C}^2/\mathbb{Z}_M$ .  $X$  is hyper-Kahler with a triple of Kahler forms  $\vec{\omega}$ ;  $X$  has  $M - 1$  non-trivial two-cycles  $\Sigma_i$ ,  $i = 1 \dots M - 1$ , which generate  $H_2$ , and

$$\int_{\Sigma_i} \vec{\omega} = \vec{\zeta}_i, \quad (3.1)$$

with  $\vec{\zeta}_i$  the  $M - 1$  blowing up parameters. Because  $\pi_1(X_\infty) = \mathbb{Z}_M$ , there can be group elements  $\rho_\infty \in U(N)$  at infinity representing  $\mathbb{Z}_M$ :  $\rho_\infty$  has  $w_\mu$  eigenvalues  $e^{2\pi i \mu/M}$ ,  $\mu = 0 \dots M-1$ , with  $w_\mu n_\mu = N$ . We define  $n_\mu \equiv 1$ , the  $SU(M)$  Dynkin indices; all greek indices run from  $0 \dots M - 1$ , with repeated indices summed.  $\rho_\infty$  breaks  $U(N) \rightarrow \prod_{\mu=0}^r U(w_\mu)$ .

In addition, because  $X$  has non-trivial two cycles  $\Sigma_i$ ,  $i = 1 \dots M - 1$ , there can also be non-trivial first Chern classes  $u_i \equiv \int_{\Sigma_i} \text{tr } F$ . We note that

$$u_i = N \int_{\Sigma_i} B, \quad (3.2)$$

where  $B$  is the usual NSNS two-form  $B_{\mu\nu}$  field. The relation (3.2) follows from the mechanism of [21], which was applied to  $D$  strings in [19]: in the presence of the Dirichlet boundary conditions, the world sheet  $B$  field mixes with the field strength of the gauge field which couples to the Dirichlet boundary. The gauge invariant field strength two-form for the overall  $U(1)$  gauge field coupling to the boundary is  $\mathcal{F} = N^{-1} \text{tr } F - B$ , where the  $N^{-1}$  accounts for the trace. Because the two-cycles  $\Sigma_i$  are closed,  $\int_{\Sigma_i} \mathcal{F} = 0$ , which gives (3.2). It is only the NSNS  $B$  field, which couples to winding, which mixes with the gauge field strength; the RR  $B$  field does not enter in  $\mathcal{F}$ . Thus in type I, where the NSNS  $B$  field is projected out, there is no such mechanism for getting non-trivial first Chern classes.

The moduli space  $\mathcal{M}_{Inst}$  of  $U(N)$  instantons on arbitrary ALE spaces was given a hyper-Kahler quotient construction in [5]; i.e.  $\mathcal{M}_{Inst}$  is isomorphic to the Higgs branch of a supersymmetric gauge theory with eight super-charges. For our case of  $\mathbb{C}^2/\mathbb{Z}_M$ , the gauge group is  $\prod_{\mu=0}^{M-1} U(v_\mu)$  with matter in the  $\oplus_\mu w_\mu \square_\mu$  and  $\oplus_\mu (\square_\mu, \square_{\mu+1})$ , where the subscript labels the gauge group and  $\square_M \equiv \square_0$ . The numbers of colors,  $v_\mu$ , are related to the physical data  $w_\mu$  and  $u_i$  discussed above via [5]

$$\tilde{C}_{\mu\nu} v_\nu = w_\mu - u_\mu. \quad (3.3)$$

$\tilde{C}_{\mu\nu}$  is the extended Cartan matrix for  $SU(M)$  and  $u_0$  is determined by  $N \equiv w_\mu n_\mu = u_\mu n_\mu$ , which follows from (3.3) and  $\tilde{C}_{\mu\nu} n_\mu = 0$ . The solution of (3.3) is

$$v_\mu = K + G_{\mu\nu}(w_\nu - u_\nu), \quad (3.4)$$

where  $K$  is an arbitrary non-negative integer and  $G_{\mu\nu} = G_{\nu\mu}$  is defined by  $G_{\mu < \nu} \equiv \mu(M - \nu)/M$ .

### 3.2. The orbifold calculation

It was shown in [4] that the gauge theories of [5], reviewed in the previous subsection, arise physically in the world-volume of type II branes at the orbifold singularities. In [4] the tadpole equations were not imposed and the physical restriction (3.3) on the  $v_\mu$  was not obtained. As discussed in the previous section, however, the twisted sector tadpoles should constrain the gauge group. We show that they indeed yield (3.3) with, via (3.2), a specific value for the  $B$  field.

Following the discussion in [4],  $\gamma_g$  will represent the action of the generator,  $g$ , of  $\mathbb{Z}_M$  on the Chan-Paton factors, thus

$$\gamma_g^M = 1. \quad (3.5)$$

We can always choose a  $\gamma_g$  satisfying (3.5) to be a diagonal matrix of dimension  $w \cdot n$  (dimension  $v \cdot n$ ) for  $p + 4$  ( $p$ ) branes such that  $w_\mu$  ( $v_\mu$ ) components along the diagonal are  $\alpha_\mu = e^{\frac{2\pi i \mu}{M}}$ . As shown in [4], the orbifold indeed yields a gauge theory with  $\prod_\mu U(v_\mu)$  gauge group and matter content discussed above.

The twisted tadpoles calculated by [10] are directly applicable for type II instantons on the  $\mathbb{Z}_M$  orbifold: we simply drop their cross-cap terms. For every  $k$  in  $1 \leq k \leq M - 1$ , the following tadpole equations must then be satisfied to avoid a divergence (for finite  $\mathcal{V}_{5-p}$ ):

$$\frac{1}{4 \sin^2(\frac{\pi k}{M})} \left( \sum_{\mu=0}^{M-1} w_\mu e^{\frac{2\pi i k \mu}{M}} - 4 \sin^2\left(\frac{\pi k}{M}\right) v_\mu e^{\frac{2\pi i k \mu}{M}} \right)^2 = 0. \quad (3.6)$$

A small manipulation converts these conditions to exactly the conditions (3.3), though with a specific value of the first Chern classes:  $u_\mu = N n_\mu / M$ . It follows from (3.2) that

$$\int_{\Sigma_i} B = \frac{1}{M} \quad (3.7)$$

for all  $i = 1 \dots M - 1$ . As in [22], in the orbifold construction of the theory on  $\mathbb{C}^2/\mathbb{Z}_M$ , there is thus a fixed non-zero value of the  $B$  field turned on. The result (3.7) for the value of the  $B$  field was also obtained in [17] by requiring  $D0$  branes to have the correct mass.

## 4. Type I or heterotic – space-time analysis

We now consider type I or heterotic small  $Spin(32)$  instanton  $D5$  branes sitting at the  $\mathbb{C}^2/\mathbb{Z}_M$  singularity, reviewing the space-time analysis of [6]. As in the previous section,

in addition to the number  $K$  of  $D5$  branes, it is necessary to specify  $\rho_\infty \in Spin(32)$  representing  $\mathbb{Z}_M$ . Following the discussion in [23], the gauge group in type I or heterotic is actually  $Spin(32)/\mathbb{Z}_2$ , where the  $\mathbb{Z}_2$  is generated by the element  $w$  in the center of  $Spin(32)$  which acts as  $-1$  on the vector,  $-1$  on the spinor of negative chirality, and  $+1$  on the spinor of positive chirality. Because only representations with  $w = 1$  are in the  $Spin(32)/\mathbb{Z}_2$  string theory, the identity element  $e \in \mathbb{Z}_M$  can be mapped to either the element  $1$  or  $w$  in  $Spin(32)$ . Thus either  $\rho_\infty^M = 1$ , which will be referred to as the case with possible<sup>3</sup> vector structure, or  $\rho_\infty^M = w$ , which will be referred to as the case without vector structure.

#### 4.1. Case $\rho_\infty^M = 1$ with possible vector structure

The group element  $\rho_\infty \in Spin(32)$  which satisfies  $\rho_\infty^M = 1$  has  $w_\mu$  eigenvalues  $e^{2\pi i \mu/M}$ ,  $\mu = 0 \dots M-1$ , with  $w_\mu n_\mu = 32$  and  $w_\mu = w_{M-\mu}$  in order to have  $\rho_\infty \in Spin(32)$ ;  $\rho_\infty$  breaks  $Spin(32) \rightarrow Spin(w_0) \times U(w_1) \times \dots \times U(w_{P-1}) \times Spin(w_P)$  for  $M = 2P$  and  $Spin(32) \rightarrow Spin(w_1) \times U(w_1) \times \dots \times U(w_P)$  for  $M = 2P+1$ .

The relevant small instanton gauge theory for  $M = 2P$  is

$$Sp(v_0) \times U(v_1) \times U(v_2) \times \dots \times U(v_{P-1}) \times Sp(v_P), \quad M = 2P, \quad (4.1)$$

with hypermultiplets  $\frac{1}{2}w_0 \cdot \square_0$ ,  $\oplus_{j=1}^{P-1} w_j \cdot \square_j$ ,  $\frac{1}{2}w_P \cdot \square_P$ , and  $\oplus_{j=1}^P (\square_{j-1}, \square_j)$  (subscripts label the gauge group). For  $M = 2P+1$  the relevant gauge theory is

$$Sp(v_0) \times U(v_1) \times U(v_2) \times \dots \times U(v_{P-1}) \times U(v_P), \quad M = 2P+1, \quad (4.2)$$

with hypermultiplets in the  $\frac{1}{2}w_0 \cdot \square_0$ ,  $\oplus_{j=1}^P w_j \cdot \square_j$ ,  $\oplus_{j=1}^P (\square_{j-1}, \square_j)$ , and  $\square_P$ . The theories (4.1) and (4.2) are given, respectively, by the “I.4” and “I.2” quiver diagrams described in sect. 4.4 of [4]. They have the unbroken space-time gauge symmetry as global symmetries.

The theories (4.1) and (4.2) have a non-trivial six dimensional gauge anomaly:

$$\mathcal{A} = \frac{1}{2} \sum_{\mu=0}^{M-1} (\tilde{C}_{\mu\nu} V_\nu - w_\mu + D_\mu) \text{tr} F_\mu^4 + 3 \sum_{j=1}^P (\text{tr} F_{j-1}^2 - \text{tr} F_j^2)^2; \quad (4.3)$$

$F_\mu$  for  $\mu > P$  is defined by  $F_\mu \equiv F_{M-\mu}$ ,  $\tilde{C}_{\mu\nu} \equiv 2\delta_{\mu\nu} - a_{\mu\nu}$  is the Cartan matrix of the extended  $SU(M)$  Dynkin diagram, and  $D_\mu \equiv 8(2\delta_{\mu,0} + \delta_{\mu,P} + \delta_{\mu,M-P})$ . We define  $V_\mu$  for

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<sup>3</sup> The “possible” qualifier is because the notion of vector structure is actually ill-defined on the Coulomb branch [24].



$\mu = 0 \dots M - 1$  by  $V_{\mu \leq P} \equiv \dim(\square_\mu)$  and  $V_{\mu > P} \equiv V_{M-\mu}$ ; i.e.  $V_0 \equiv 2v_0$ ,  $V_{i < P} \equiv v_i$ , and  $V_P \equiv 2v_P$  ( $V_P \equiv v_P$ ) for  $M = 2P$  ( $M = 2P + 1$ ). Also, as the theories (4.1) and (4.2) only have  $M - P - 1$   $U(1)$  factors to which Fayet-Iliopoulos terms can be coupled, they are missing  $P$  hypermultiplet blowing up moduli.

It was conjectured in [6] that there is a ‘‘Coulomb branch’’ phase of the heterotic or type I string theory where the theory in the world volume of the D5 branes is of the form (4.1) or (4.2), but with  $29P$  hypermultiplets traded for  $P$  additional tensor multiplets. In order to cancel the irreducible  $\text{tr} F_\mu^4$  part of the gauge anomaly (4.3), it is necessary to have

$$\tilde{C}_{\mu\nu} V_\nu = w_\mu - D_\mu; \quad (4.4)$$

this gives  $w_\mu n_\mu = 32$  as expected. The solution of (4.4) is

$$V_{\mu \leq P} = 2K + \sum_{\nu=0}^P \min(\mu, \nu) W_\nu - 8\mu, \quad (4.5)$$

with  $K$  an arbitrary positive integer (which is large enough so that all  $V_\mu \geq 0$ ) and  $W_{\nu < P} \equiv w_\mu$  and  $W_P \equiv w_P$  ( $W_P \equiv \frac{1}{2}w_P$ ) for  $M$  odd (even). With the relation (4.4), there are  $28P$  missing hypermultiplets associated with deforming the instanton moduli; remembering the  $P$  missing blowing up moduli mentioned above, there are  $29P$  missing hypermultiplets. Finally, in order to cancel the reducible  $\text{tr} F_\mu^2 \text{tr} F_\nu^2$  anomaly in (4.3), it is necessary to introduce  $P$  tensor multiplets with couplings to the gauge fields in (4.1) or (4.2) which are the supersymmetric completion of the interactions

$$\sum_{i=0}^P (\Phi_{i+1} - \Phi_i) \text{tr} F_i^2, \quad (4.6)$$

where  $\Phi_i$  are the real scalar components of the tensor multiplets, with  $\Phi_0 \equiv \Phi_{P+1} \equiv 0$ .

#### 4.2. Case $\rho_\infty^M = w$ without vector structure

This case requires  $M = 2P$ ;  $\rho_\infty$  has  $w_j$  eigenvalues  $e^{i\pi(2j-1)/2P}$ ,  $j = 1 \dots 2P$ , with  $w_{2P+1-j} = w_j$  and  $\sum_{j=1}^{2P} w_j = 2 \sum_{j=1}^P w_j = 32$  for  $\rho_\infty \in Spin(32)$ .

It was conjectured in [6] that the world-volume theory has a ‘‘Coulomb branch’’ phase with  $29(P - 1)$  hypermultiplets traded for  $P - 1$  tensor multiplets. The gauge theory on

the Coulomb branch is  $\prod_{i=1}^P U(v_i)$  with matter content  $\oplus_{i=1}^P w_i \cdot \square_i$ ,  $\oplus_{i=1}^{P-1} (\square_i, \square_{i+1})$ ,  $\square_1$ , and  $\square_P$ . This is described by the “type I5” quiver diagrams of [4]. The gauge anomaly is

$$\mathcal{A} = \frac{1}{2} \sum_{i=1}^{2P} \left( \sum_{j=1}^{2P} \tilde{C}_{ij} v_j - w_i + D_i \right) \text{tr} F_i^4 + 3 \sum_{r=1}^{P-1} (\text{tr} F_r^2 - \text{tr} F_{r+1}^2)^2, \quad (4.7)$$

where  $\tilde{C}_{ij}$  is the Cartan matrix for the extended  $SU(2P)$  Dynkin diagram,  $F_{i>P} \equiv F_{2P+1-i}$ ,  $v_{i>P} \equiv v_{2P+1-i}$ , and  $D_i \equiv 8(\delta_{i,1} + \delta_{i,P} + \delta_{i,2P} + \delta_{i,P+1})$ . Thus, in order to cancel the irreducible  $\text{tr} F_i^4$  gauge anomaly, it is necessary to have

$$\sum_{j=1}^{2P} \tilde{C}_{ij} v_j = w_i - D_i; \quad (4.8)$$

note that this properly gives  $w_\mu n_\mu = 32$ . The solution of (4.8) is

$$v_{i \leq P} = 2K + \sum_{j=1}^P \min(i-1, j-1) w_j - 8(i-1), \quad (4.9)$$

where  $K$  is an arbitrary non-negative integer (which is large enough so that all  $v_i \geq 0$ ).

In order to cancel the reducible gauge anomaly in (4.7), it is necessary to have  $P-1$  tensor multiplets with coupling to the gauge fields  $F_i$  of  $U(v_i)$  which are the supersymmetric completion of

$$\sum_{i=1}^P (\Phi_i - \Phi_{i-1}) \text{tr} F_i^2, \quad (4.10)$$

where  $\Phi_i$  are the scalar components of the tensor multiplets, with  $\Phi_0 \equiv \Phi_P \equiv 0$ .

## 5. The orientifold analysis

In this section, we derive via a direct orientifold analysis the “Coulomb branch” phases [6] reviewed in the previous section. As the relevant gauge theories were already obtained in [4] by an orientifold construction, all that remains to do is to: explain why there are the extra tensor multiplets and missing hyper-multiplets, obtain the relations (4.4) and (4.8), and obtain the couplings (4.6) and (4.10).

In the following we call  $\Omega$  the element that reverses the orientation of a type IIB string and  $\gamma_\Omega$  the matrix representing the action of  $\Omega$  on Chan-Paton factors. In addition to the condition (3.5), there are two more algebraic consistency conditions [3,4]:

$$\gamma_\Omega = \pm \gamma_\Omega^T \quad (5.1)$$

where the upper(lower) sign is for 9(5) branes, and

$$\gamma_g \gamma_\Omega \gamma_g^T = \alpha \gamma_\Omega. \quad (5.2)$$

The case without vector structure will always have  $\alpha = \alpha_1$  and can only occur for  $M$  even, while the case with possible vector structure will have  $\alpha = \alpha_0 = 1$ .

The above two conditions imply that for  $M$  even there are two possible actions of  $\Omega$  on Chan-Paton factors; they correspond to with or without possible vector structure. In the convention of [10], the action of  $\Omega$  on the Chan-Paton factors for these two cases is such that

$$\text{tr}(\gamma_{\Omega k}^{-1} \gamma_{\Omega k}^T) = \epsilon \text{tr}(\gamma_{\Omega k+M/2}^{-1} \gamma_{\Omega k+M/2}^T) \quad (5.3)$$

where  $\epsilon = +1(-1)$  for the case with (without) vector structure. The two possible actions of  $\gamma_\Omega$  correspond to two different actions of  $\Omega$  on the world-sheet. Some of the following is similar to a discussion of [12]. There is only one choice for the action of  $\Omega$  in the untwisted sector that will not project out the graviton. In each of the other twisted sectors we have *a priori* two choices for  $\Omega$ , one of which will preserve a tensor multiplet and the other of which will preserve a hypermultiplet. We will show that tree-loop duality of the non-oriented and open string amplitudes reduces this choice to the  $\mathbb{Z}_2$  twisted sector. The case without vector structure will preserve the tensor while the other case will preserve the hypermultiplet. Given the choice of  $\Omega$ , we fix  $\epsilon$  and the  $\mathbb{Z}_2$  twisted sector of the loop Klein bottle. This choice, in turn, fixes the signs of the Mobius strip amplitudes. The relative signs of all cross-caps are then fixed by the duality relation. Since we can only obtain two possibilities for consistent tadpole amplitudes, this means that the sign of  $\Omega$  is relatively fixed in the other twisted sectors. What is the sign of  $\Omega$ ? Let us suppose that we have a closed bosonic string state twisted by  $g^k$  propagating around a cylinder. Now apply  $\Omega$  and close the cylinder to create a Klein bottle. We know this amplitude vanishes by loop-tree duality for  $k \neq 0, M/2$ . This implies that  $\Omega$  acts in one of two ways. It either switches the  $k$  twist to the  $M - k$  twist (in other words it includes  $J$  [13]), which then vanishes in the trace, or  $\Omega$  acts with a different sign to the left than to the right so that the amplitude vanishes. The second choice is equivalent to  $\Omega$  having a relative sign between the  $k$  twisted and  $M - k$  twisted sectors for  $k \neq 0, M/2$ , and we will assume corresponds to the case with vector structure ( $\epsilon = +1$ ).  $J$  corresponds to the case without vector structure ( $\epsilon = -1$ ).

Both of the above choices for the action of  $\Omega$  mean that the twisted sectors with  $k \neq 0, \frac{1}{2}M$  yield  $\frac{M}{2} - 1$  ( $\frac{M-1}{2}$ ) extra tensors for  $M$  even (odd), along with the same

number of hyper-multiplets. These hyper-multiplets transform, as in the discussion in [4,23], to cancel the anomalies associated with the  $U(1)$  factors in the gauge groups. They pair up with the  $U(1)$  gauge fields to give them a mass and their expectation values become Fayet-Iliopoulos parameters. Because the above argument for the presence of extra tensor multiplets is quite general, orientifolds can never realize standard type I theories at ALE singularities.

The twisted tadpoles exactly reproduce (up to constants) the individual terms in the space-time anomalies. Our general procedure will be as follows. The twisted tadpoles will be of the form  $\sum_{g \neq 1} (a_\mu^g w_\mu + b_\mu^g v_\mu + c^g)^2$ . To obtain the irreducible  $\text{tr}(F_\mu)^4$  term in the anomaly, note that this term in the tadpole should be proportional to  $\text{tr}(I_{v_\mu}) = v_\mu$  ( $I_n$  is the  $n \times n$  dimensional identity matrix), and this term will thus be proportional to  $\sum_{g \neq 1} b_\mu^g (a_\nu^g w_\nu + b_\nu^g v_\nu + c^g)$ . Similarly, to obtain the reducible  $\text{tr}(F_\mu)^2 \text{tr}(F_\nu)^2$  term, we extract the coefficient of  $v_\mu v_\nu$ , which will be  $\sum_{g \neq 1} b_\mu^g b_\nu^g$ . The other pieces of the tadpole are the reduction of the  $p + 5$  dimensional gauge anomaly to  $p + 1$  dimensions and the gravitational anomaly. In fact, the irreducible gravitational anomaly should have the coefficient  $\sum_{g \neq 1} c^g (a_\mu^g w_\mu + b_\mu^g v_\mu + c^g)$ . (For type IIB,  $c^g = 0$  so there is no  $p + 1$  dimensional gravitational anomaly.)

From the reducible term, we obtain the coefficient (up to a constant) of the twisted sector  $B$  fields coupling to the gauge fields. The same  $B$  fields also couples to the nine-brane gauge fields with the coefficient of  $w_\mu w_\nu$ <sup>4</sup>. The tensor  $B$  components of the tensor multiplet come from the twisted Ramond-Ramond sector while the real scalar comes from the twisted NS-NS sector; they have the same couplings, as is required by supersymmetry, since it is the same tadpole for NS-NS and R-R.

### 5.1. Case with possible vector structure

To derive the above results from the orientifold, we augment the results of [4] with those of [10]. In the closed string calculation, we must assume that an extra minus exists in the action of  $\Omega$  for sectors twisted by  $1 - t$  relative to those twisted by  $t$  for  $0 < t < \frac{1}{2}$ . The extra minus is also present for  $t = \frac{1}{2}$ . In sectors with the extra minus, the three-brane

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<sup>4</sup> When combined with the fact that  $\int_{ALE} dH \neq 0$  with five-branes present, this means that there is an anomaly inflow mechanism, with net current flowing onto the branes from the ten-dimensional world. This is the limit of the mechanism of [25] where the instantons shrink to zero size.

is not projected out, and one obtains a tensor multiplet rather than a hypermultiplet. This extra minus changes the sign of terms twisted by  $\mathbb{Z}_2$  in the one loop Klein bottle so that the tadpoles of [10] are modified. Using the solutions of  $\gamma_\Omega$  and  $\gamma_g$  derived by [4], we obtain  $w \cdot n = 32$  and

$$\frac{1}{4 \sin^2(\frac{\pi k}{M})} \left( \sum_{\mu=0}^{M-1} w_\mu e^{\frac{2\pi i k \mu}{M}} - 4 \sin^2(\frac{\pi k}{M}) v_\mu e^{\frac{2\pi i k \mu}{M}} - 32 \delta_{k,0 \bmod 2} \right)^2 = 0 \quad (5.4)$$

for  $M$  even, and

$$\frac{1}{4 \sin^2(\frac{2\pi k}{M})} \left( \sum_{\mu=0}^{M-1} w_\mu e^{\frac{4\pi i k \mu}{M}} - 4 \sin^2(\frac{2\pi k}{M}) v_\mu e^{\frac{4\pi i k \mu}{M}} - 32 \cos^2(\frac{\pi k}{M}) \right)^2 = 0 \quad (5.5)$$

for  $M$  odd. Using the method discussed above for extracting the appropriate terms, these tadpoles yield exactly the space-time anomaly (4.3) (up to the relative factor of 6 between the reducible and irreducible parts).

To summarize, the orientifold analysis gives exactly the ‘‘Coulomb phase’’ spectrum and tensor multiplet couplings of [6]. For the  $\mathbb{Z}_2$  case, the Coulomb phase had previously been constructed via  $F$  theory in the context of a compact  $K3$  (thus there was an upper bound on the possible  $K$ ) [24].

## 5.2. Case without vector structure

In the orientifold calculation, the extra minus in the  $\mathbb{Z}_2$  twisted sector is no longer present. Thus, in the  $\mathbb{Z}_2$  twisted closed string sector, the three-brane is now projected out, and we obtain a hypermultiplet rather than a tensor, giving one less tensor as compared to the possible vector structure case; there are then a total of  $P-1$  extra tensor multiplets. To get the anomalies we use the  $\gamma_\Omega$  and  $\gamma_g$  determined by the algebraic consistency conditions of [4] and the tadpoles of [10]. Note that there is an extra phase in the  $\gamma$  matrices of [4] as compared to [10], i.e.,  $\text{tr}(\gamma_{\Omega k,9}^{-1} \gamma_{\Omega k,9}^T) = e^{-2\pi i k/M} \text{tr}(\gamma_{2k,9})$ , etc. The extra minus in the Klein bottle  $\mathbb{Z}_2$  twisted sector goes away, and we directly apply the tadpoles of [10] with  $w \cdot n = 32$  and

$$\frac{1}{4 \sin^2(\frac{\pi k}{M})} \left( \sum_{\mu=0}^{M-1} w_\mu e^{\frac{2\pi i k \mu}{M}} - 4 \sin^2(\frac{\pi k}{M}) v_\mu e^{\frac{2\pi i k \mu}{M}} - 16 \delta_{k,0 \bmod 2} (1 + e^{4\pi i k/M}) \right)^2 = 0. \quad (5.6)$$

Again these results give (4.7) after a brief calculation. Note that for  $M$  odd, there are other theories with 0 nine-branes, one of which was discussed by [10]. In this case one expects two possible projections of  $\Omega$ , one of which gives twisted strings.

## 6. Interpretation of the extra tensor multiplets

The above, perturbative, orientifold analysis applies far out along the “Coulomb branch,” where the tensors have large expectation value and the theory is weakly coupled. Let us try to obtain an intuitive picture for how the extra tensors of the Coulomb branch arise in type I orientifolds. In the context of type IIB at the ALE singularity, there is a moduli space with  $r$   $\mathcal{N} = (2, 0)$  matter multiplets, each consisting of a  $\mathcal{N} = (1, 0)$  tensor multiplet plus hypermultiplet, giving  $5r$  real scalars. The  $4r$  scalars arising from the  $\mathcal{N} = (1, 0)$  hypermultiplets have expectation values determined by the  $3r$  blowing up parameters (3.1) and  $B$  fields (3.2), while the expectation values of the  $r$  scalars from the  $\mathcal{N} = (1, 0)$  tensor multiplets are more subtle: as in [26,27,28], they can be interpreted geometrically as the separations of five-branes in the extra dimension of  $M$  theory. There are strings, arising from wrapping three-branes on the cycles  $\Sigma_i$ , which couple to the tensor multiplet and become tensionless where the tensor multiplet has zero expectation value.

Upon orientifolding to obtain type I, one either keeps the  $\mathcal{N} = (1, 0)$  hypermultiplet part of the  $\mathcal{N} = (2, 0)$  matter multiplet or one keeps the  $\mathcal{N} = (1, 0)$  tensor multiplet. We expect, following the above discussion, that the expectation values of the tensor multiplets which we obtained should have an interpretation as distances of five-branes in the extra direction of  $M$  theory. Consider, for example  $M$  theory on  $\mathbb{R}^1/\mathbb{Z}_2$ . Arguments similar to [29], suggest that the theory at the origin of  $\mathbb{R}^1/\mathbb{Z}_2$  is the infinite coupling limit of the type I  $SO(32)$  ten-dimensional string theory. By “compactifying” this theory on the infinite volume ALE space, we can obtain a finitely coupled type I theory. (We would need another dimension, as in  $F$  theory, to obtain a finitely coupled type I theory in ten dimensions.) In analogy with the  $E_8$  case, we expect that a small  $SO(32)$  instanton turns into an NS five-brane which can wander away from the origin into  $\mathbb{R}/\mathbb{Z}_2$ , with distance equal to the expectation value of the tensor multiplet.

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